A (co-)End Approach to Day Convolution

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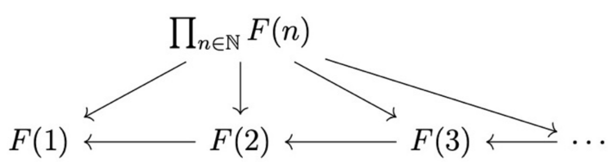
(co)Limits and (co)Ends

The most common definition of a (category-theoretic) limit, and the most intuitive perspective in many use cases, is as the universal cone over a particular diagram. We begin by unraveling this definition into a definition via equalizers. From this perspective, it becomes clear that limits are a special case of a more general construction known as *ends*. These are powerful tools- in our case, their associated calculus, the "(co)end calculus", will enable us to elegantly prove powerful statements. There is of course a dual notion of a *coend*, but for the sake of motivation we approach the (co)end calculus via limits.

Corollary 1.1. The limit object of a presheaf F is the set of all cones with respect to F.

We begin with the following approach in Mac Lane's "Categories for the Working Mathematician" (p. 105). Consider a presheaf F over the category of ordinals $\mathbb{N}$, whose morphisms are the standard ordering. Then we have the following diagram



(The morphisms are reversed because the source of F is the opposite category of $\mathbb{N}$.) But as it is, this diagram may not necessarily commute, i.e., not every triplet may constitute a cone. (The above diagram only indicated the source and targets of maps.) From this we are led to consider cones as specific "pieces" of the maps

$$\prod_{n\in\mathbb{N}} F(n) \to \prod_{n\in\mathbb{N}} F(n).$$

We want to express the limit in terms of this map, but first we will generalize. For a general presheaf $C^{\mathrm{op}} \to \mathrm{Set}$, the codomain we are actually interested in is the product of all images of morphisms in C^{op}. For otherwise there couldn't possibly be a cone. We adopt the convention that given a morphism f, write $s(f)$ for the source object and $t(f)$ for the target object. We can then express the above more generally as a map

$$\prod_{c\in C^{\mathrm{op}}} F(c) \to \prod_{f\in\mathrm{Mor}(C^{\mathrm{op}})} F(t(f)).$$

Recall that a cone is a collection of maps p_j for each $j \in C^{\mathrm{op}}$ such that for all morphisms $f: j \to k$ the following diagram commutes:

$$\begin{array}{ccc} & A & \\ {}^{p_j}\swarrow & & \searrow{}^{p_k} \\ F(j) & \xrightarrow{\quad F(f) \quad} & F(k) \end{array}$$

(Here, $A \in \mathrm{Set}$.) This is the condition that $F(f) \circ p_j = p_k$. By the definition of a product, every map p_j factors uniquely through $\prod_j F(j) \to A$. We can consider the maps p_j as maps out of the product, rather than out of A. (The maps out of the product are the canonical projections. At the risk of abusing notation for transparency, we will also call these maps p_j.) We can thus consider the requirement that $F(f) \circ p_j = p_k$ as defining the equalizer of the parallel morphisms $F(f) \circ p_j$ and p_k. This is by definition the set of all cones, hence the limit of the presheaf F (cor. 1):

$$\lim F \longrightarrow \prod_{j \in C^{\mathrm{op}}} F(j) \underset{\prod_{f \in \mathrm{Mor}(j)} p_{t(f)}}{\overset{\prod_{f \in \mathrm{Mor}(j)} F(f) \circ p_j}{\rightrightarrows}} \prod_{f \in \mathrm{Mor}(j)} F(t(f))$$

(By writing $\mathrm{Mor}(j)$ we mean the set of morphisms in the ambient category that j belongs to which have j as a source.) This is the approach taken in the nLab *limit* article.

In enriched category theory, the set of morphisms is an object (not just a set, for instance) in the category we enriching over. So we are motivated to rework the equalizer above- it no longer makes sense to index the products over the *set* morphisms; we need to somehow index over the *space* of morphisms.

It makes sense to start by reworking the codomain. Notice that

$$\prod_{f \in \mathrm{Mor}(j)} F(t(f)) \cong \prod_{j,k} F(k)^{C^{\mathrm{op}}(j,k)}$$

where

$$F(k)^{C^{\mathrm{op}}(j,k)} = [C^{\mathrm{op}}(j,k), F(k)].$$

Here we consider $C^{\mathrm{op}}(j,k)$ as a set, and so $F(k)^{C^{\mathrm{op}}(j,k)}$ as a set has precisely as many elements as there are morphisms in C^{op} from j to k. The upshot of this phrasing is that it suggests a generalization to the case where C^{op} is V-enriched (hence our notation). For then $C^{\mathrm{op}}(j,k) \in V$, and, if we view V as enriched over itself, $F(k)^{C^{\mathrm{op}}(j,k)} \in V$ as well.

Now we need to rework the maps we are equalizing. First let's consider the map $F(f) \circ p_j$. There isn't anything wrong with the projection map, even in our new codomain we still have a projection map, though it is of course defined slightly differently. (In particular by postcomposing with the isomorphism above.) At the risk of abusing notation for the sake of transparency, we will still call the projection p_j. We do, however, need to redefine $F(f)$ in light of our new codomain. Observe that $F(f)$ is equivalently a map

$$(j \to k) \quad \mapsto \quad (F(j) \to F(k)).$$

The adjunct of this map

$$\lambda_{j,k}: F(j) \to [C^{\mathrm{op}}(j,k), F(k)]$$

is of the form we desire, and captures the same information. Hence the analog to the map $\prod_{f \in \mathrm{Mor}(j)} F(f) \circ p_j$ is

$$\lambda := \prod_{j,k \in C^{\mathrm{op}}} \lambda_{j,k} \circ p_j.$$

Now let's consider the other map we are equalizing, p_k. This map contains the information of the projection, as well as only being defined for k that were targets of morphisms in C^{op}. The set of k that are targets of morphisms are in bijection with maps of the form

$$(j \to k) \to * \to (F(k) \to F(k)).$$

(Here, the second map is the adjunct of the identity on $F(k)$, hence our factoring through $*$ serves to indicate we are sending $j \to k$ to the identity on $F(k)$, which is necessary for our bijection.) The adjunct of this map

$$\rho_{j,k}: F(k) \to [C^{\mathrm{op}}(j,k), F(k)]$$

is of the form we want and contains the same information. Hence the analog to the map $\prod_{f \in \mathrm{Mor}(j)} p_{t(f)}$ is

$$\rho := \prod_{j,k \in D^{\mathrm{op}}} \rho_{j,k} \circ p_k.$$

A limit is thus the equalizer

$$\lim F \longrightarrow \prod_{j \in C^{\mathrm{op}}} F(j) \underset{\rho}{\overset{\lambda}{\rightrightarrows}} \prod_{j,k \in C^{\mathrm{op}}} [C^{\mathrm{op}}(j,k), F(k)]$$

But observe that given a pair (j,k), we defined $\lambda_{j,k}$ to have source $F(j)$ and $\rho_{j,k}$ to have source $F(k)$. By having the domain of the parallel morphisms being indexed only over the product all $F(j)$, we are only really considering the j which are both the source and target of at least one (perhaps seperate) morphism. We thus manifest *ends* as a generalization of limits that preserves this nuance, following the *ends* article from the nLab:

Definition 2. For V a symmetric monoidal category, C a V-enriched category and $F: C^{\mathrm{op}} \times C \to V$ a V-enriched functor, the *end* of F is the equalizer

$$\int_{c \in C} F(c,c) \longrightarrow \prod_{c \in \mathrm{Obj}(C)} F(c,c) \underset{\rho}{\overset{\lambda}{\rightrightarrows}} \prod_{c_1, c_2 \in \mathrm{Obj}(C)} [C(c_1,c_2), F(c_1,c_2)]$$

with ρ given in components by

$$\rho_{c_1,c_2}: F(c_1, c_1) \to [C(c_1, c_2), F(c_1, c_2)]$$

being the adjunct of

$$F(c_1, -): C(c_1, c_2) \to [F(c_1, c_1), F(c_1, c_2)]$$

and

$$\lambda_{c_1,c_2}: F(c_2, c_2) \to [C(c_1, c_2), F(c_1, c_2)]$$

being the adjunct of

$$F(-, c_2): C(c_1, c_2) \to [F(c_2, c_2), F(c_1, c_2)].$$

Dually, we define

Definition 3. The *coend* of F is the coequalizer

$$\coprod_{c_1,c_2} C(c_2, c_1) \otimes F(c_1, c_2) \rightrightarrows \coprod_{c} F(c, c) \longrightarrow \int^{c} F(c, c)$$

with the parallel morphisms induced by the two actions of F.

Remark 4. Typically, enriching over a "sufficiently nice" category V means requiring V to be symmetric monoidal. If we further assume V is closed monoidal, then V can be considered enriched over itself. Hence forth, we will assume V to by symmetric monoidal.

Example 5. In the above, F is a functor from $C^{\text{op}} \times C \to V$. Thus, if we define F such that $F(c_1, c_2) = F'(c_1)$ for some functor $F': C^{\text{op}} \to V$, we recover the notions of a limit and colimit:

$$\lim F = \int_{c \in C} F'(c) := \int_{c \in C} F(c, c),$$
$$\operatorname{colim} F = \int^{c \in C} F'(c) := \int^{c \in C} F(c, c).$$

In the spirit of consistency, limits and colimits are often written in this notation when (co)ends are involved.

Example 6. What is almost immediate after unraveling definitions is the expression of natural transformations in terms of ends: given $F, G: C \to D$,

$$[C, D](F, G) = \int_{c \in C} D(F(c), G(c)).$$

Taking the end, this is the collection of $F(c) \to G(c)$ that are in the equalizer as above, i.e. satisfy that for each $c_1 \to c_2$ there exists a commuting square of the form

$$\begin{array}{ccc} F(c_1) & \longrightarrow & F(c_2) \\ \downarrow & & \downarrow \\ G(c_1) & \longrightarrow & G(c_2) \end{array} .$$

But this is equivalently a natural transformation. Since ends are universal with this property, each commuting square of the above form is in the equalizer and hence in the end.

Example 7. Given a Hom functor that preserves limits in each variable separately, we have the following properties:

$$\mathrm{Hom}(X, \int_c F\,(c,c)) \cong \int_c \mathrm{Hom}\,(X, F(c,c)),$$
$$\mathrm{Hom}(\int^c F\,(c,c), Y) \cong \int_c \mathrm{Hom}\,(F(c,c), Y).$$

Notable examples of Hom functors that preserve limits are the standard Set-valued Hom and the pointed topological mapping space

$$\mathrm{Maps}(-,-)_*: \mathrm{Top}_{\mathrm{cg}}^{*/} \times \mathrm{Top}_{\mathrm{cg}}^{*/} \to \mathrm{Top}_{\mathrm{cg}}^{*/}.$$

Example 8. The way of writing F hopefully makes transparent the idea of the end as the equalizer of the left and right actions encoded in F.

The (co)End Calculus

We now collect several results that allow us to prove things using (co)ends. These and similarly motivated results constitute the *co(end) calculus*. We draw primarily from the *ends* and *Introduction to Stable homotopy theory -- 1-2* articles from the nLab. The particular form of Proposition 10 is due to Loregian's "(co)End Calculus", which is an excellent comprehensive reference for the general subject.

Corollary 9. Any continuous functor K preserves ends, and any cocontinuous functor K' preserves coends. For instance, given a functor $F: C^{\mathrm{op}} \times C \to V$ and $c \in C$, $v \in V$,

$$K(v, \int_c F\,(c,c)) \cong \int_c K\,(v, F(c,c)),$$
$$K'(\int^c F\,(c,c), v) \cong \int^c K'(F(c,c), v).$$

Proposition 10 (Fubini's theorem for (co)ends). Given V-enriched categories C and D, and a V-enriched functor

$$F: C^{\mathrm{op}} \times C \times D^{\mathrm{op}} \times D \to V,$$

if the (co)end of F exists, then

$$\int_{c,d} F(c,c,d,d) \cong \int_d \int_c F(c,c,d,d) \cong \int_c \int_d F(c,c,d,d).$$

Dually,

$$\int^{c,d} F(c,c,d,d) \cong \int^d \int^c F(c,c,d,d) \cong \int^c \int^d F(c,c,d,d).$$

Proposition 11. Let $F, G: C \to D$ be functors. Write $[C,D](F,G)$ for the set of natural transformations between them. Then

$$[C,D](F,G) = \int_{c \in C} D(F(c), G(c)).$$

Proposition 12. For functors $S: C^{\mathrm{op}} \to D$ and $T: C \to D$,

$$S \underset{C}{\otimes} T = \int^{c \in C} S(c) \underset{D}{\otimes} T(c).$$

Lemma 13 (co-Yoneda lemma). Every V-valued presheaf $F: C^{\mathrm{op}} \to V$ is a colimit of representable presheaves:

$$F(-) \simeq \int^{c \in C} F(c) \underset{V}{\otimes} V(-, c).$$

Proof.

$$\begin{aligned} V\left(\int^{c' \in C} C(c,c') \otimes F(c'), y\right) &\cong \int_{c' \in C} V\left(\int^{c' \in C} C(c,c') \otimes F(c'), y\right) \\ &\cong \int_{c' \in C} V(C(c,c') \otimes F(c'), y) \\ &\cong \int_{c' \in C} V(C(c,c'), V(F(c'), y)) \\ &\cong [C,V](C(c,-), V(F(-), y)) \\ &\cong V(F(c), y). \end{aligned}$$

(The first isomorphism is by prop. 11, the second is prop. 12, the third is the tensor-hom adjunction, the fifth is the observation that y makes $C(c,-)$ redundant.) The result then follows by the enriched Yoneda lemma. ■

The following result is formally dual:

Proposition 14 (Yoneda reduction). There is a natural isomorphism

$$F(c') \cong \int_{c \in C} V(C(c', c), F(c)).$$

Day Convolution

We are motivated by the following example, a viewpoint illustrated by Campbell in their "Day Convolution Intuition" answer on Math Stack Exchange.

Example 15. Given a ring R and a monoid M, we wish to extend the ring structure to $R^{(M)}$, the set of functions $M \to R$ of finite support. We require that $R \to R^{(M)}$ is a ring homomorphism, and $M \to R^{(M)}$ is a monoid homomorphism. In this way the resulting ring structure on $R^{(M)}$ becomes canonical.

Let's first consider a basis for $R^{(M)}$. We claim this is the set of e_m for $m \in M$, where $e_m(x) = 1$ if $x = m$ and is 0 otherwise (for instance, see Yu's article "Vector Space of Functions from a Set to a Vector Space").

Addition is then well defined, and it remains to find a suitable multiplicative operation $*$ that makes $R^{(M)}$ into a ring. But given that $M \to R^{(M)}$ is a monoid homomorphism sending $x \mapsto e_x$, and since we can shuffle constants around the multiplication operation, we are led to define

$$\begin{aligned} f * g &= \left(\sum_{x\in M} f(x)e_x\right) * \left(\sum_{y\in M} g(y)e_y\right) \\ &= \sum_{x,y\in M} f(x)g(y)e_x * e_y \\ &= \sum_{x,y\in M} f(x)g(y)e_{xy}. \end{aligned}$$

In references, the standard way to define $*$ is

$$f * g(m) = \sum_{xy=m} f(x)g(y).$$

But observe this is equivalent:

$$\sum_{x,y\in M} f(x)g(y)e_{xy}(m) = \sum_{xy=m} f(x)g(y).$$

We call the operation $*$ the *convolution product*.

The Day convolution we are after is the categorification of this concept.

Example 16. Rather than the set of functions from a monoid M to a ring R, we are concerned with endowing a multiplicative operation to the functors from a category C^{op} to a category V (i.e. V-valued presheaves).

By the co-Yoneda lemma (lemma 13), every V-valued presheaf $F: C^{\mathrm{op}} \to V$ is a colimit of representable presheaves:

$$F(-) \simeq \int^{c \in C} F(c) \underset{V}{\otimes} V(-, c).$$

Thus we can think of the set of such colimits as a "basis" for the presheaf category $[C^{\mathrm{op}}, V]$. Since we require $*$ to be canonically defined, we calculate

$$\begin{aligned} F * G &\simeq \left(\int^{c_1 \in C^{\mathrm{op}}} F(c_1) \underset{V}{\otimes} V(-, c_1)\right) * \left(\int^{c_2 \in C^{\mathrm{op}}} G(c_2) \underset{V}{\otimes} V(-, c_2)\right) \\ &\simeq \int^{(c_1, c_2) \in C^{\mathrm{op}} \times C^{\mathrm{op}}} F(c_1) \underset{V}{\otimes} G(c_2) \underset{V}{\otimes} V(-, c_1) * V(-, c_2) \\ &\simeq \int^{(c_1, c_2) \in C^{\mathrm{op}} \times C^{\mathrm{op}}} F(c_1) \underset{V}{\otimes} G(c_2) \underset{V}{\otimes} V(-, c_1 \underset{C}{\otimes} c_2). \end{aligned}$$

We now cover the assumptions needed to calculate as we did. First, we require C and V to be a monoidal categories with respect to some tensor products $\otimes_C$ and $\otimes_V$. We used Fubini's theorem for coends (prop. 10) and so $\otimes_V$ needs to be cocontinuous in each variable. Furthermore, that the coends exist at all is the assumption that V is cocomplete. (The assumptions on V can be summarized by saying that V is *monoidally cocomplete*.)

The following definition is more standard, appearing for instance in the *Day convolution* article on the nLab. It utilizes C rather than C^{op}, thus switching the covariant hom for a contravariant one. The ordering is also switched, which is fine as long we stay consistent.

Definition 17 (Day convolution). Let $(C, \otimes_C, 1)$ be a small V-enriched monoidal category. Then the *Day convolution tensor product* on $[C, V]$

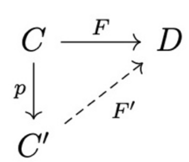
$$\underset{\mathrm{Day}}{\otimes} : [C, V] \times [C, V] \to [C, V]$$

is given by

$$X \underset{Day}{\otimes} Y : c \mapsto \int^{(c_1, c_2) \in C \times C} C(c_1 \underset{C}{\otimes} c_2, c) \underset{V}{\otimes} X(c_1) \underset{V}{\otimes} Y(c_2).$$

A Left Kan Extension

Given a functor $p: C \to C'$, it is typically interesting to understand the *extension problem*, i.e. whether any functor $F: C \to D$ can be extended to a functor $F': C' \to D$:



This is a very strong (and useful!) property to have. So even when we don't have this property, we are still interested in *approximating* the functor F', even if it means fixing a single functor F. Kan extensions seek to do just that. There are many types of Kan extensions, which all coincide in the best case scenario. For our purposes, we are interested in the following, taken from the *Kan extension* nLab article:

Definition 18 (Local Kan extensions). Given a functor $p: C \to C'$, we have an induced functor $p^*: [C', D] \to [C, D]$ given by precomposition by p.

A *left Kan extension* of F along p is then a functor $\mathrm{Lan}_p F: [C', D]$ such that

$$\mathrm{Hom}_{[C,D]}(F, p^*(-)) \cong \mathrm{Hom}_{[C',D]}(\mathrm{Lan}_p F, -)$$

is a natural isomorphism.

Perhaps more intuitively, $\mathrm{Lan}_p F$ is a corepresentation of $\mathrm{Hom}_{[C,D]}(F, p^*(-))$.

Remark 19. Def. 18 is a specific case of what are called *global Kan extensions*.

Proposition 20. The left Kan extension is given by the coend

$$(\mathrm{Lan}_p F): c' \mapsto \int^{c \in C} C'(p(c), c') \underset{D}{\otimes} F(c).$$

Proof. It suffices to show $[C', D](\mathrm{Lan}_p F, G) \cong [C, D](F, p^*(G))$. We calculate:

$$\begin{aligned}
(\mathrm{Lan}_p F, G) &= \int_{c' \in C'} D\,(\mathrm{Lan}_p F(c'), G(c')) \\
&= \int_{c' \in C'} D\left(\int^{c \in C} C'(p(c), c') \underset{D}{\otimes} F(c), G(c')\right) \\
&= \int_{c' \in C'} \int_{c \in C} D\,(C'(p(c), c') \underset{D}{\otimes} F(c), G(c')) \\
&= \int_{c \in C} \int_{c' \in C'} D\,(F(c), D(C'(p(c), c'), G(c'))) \\
&= \int_{c \in C} D\,(F(c), \int_{c' \in C'} D\,(C'(p(c), c'), G(c'))) \\
&= \int_{c \in C} D\,(F(c), \int_{c' \in C'} D\,(C'(p(c), c'), G(c'))) \\
&= [C, D](F, p^* G).
\end{aligned}$$

(The first step is ex. 6, the second step is our assumption, the third is ex. 7, the fourth is Fubini, thm. 10, along with the tensor-hom adjunction, the fifth is ex. 7 again, the sixth is Yoneda reduction, prop. 14, and the seventh is ex. 6 again.) ■

Definition 21. For C a small V-monoidal category, its external tensor product

$$\overline{\otimes}: [C, V] \times [C, V] \to [C \times C, V]$$

given by

$$X \overline{\otimes} Y := \underset{V}{\otimes} \circ (X, Y),$$
$$(X \overline{\otimes} Y)(c_1, c_2) = X(c_1) \underset{V}{\otimes} Y(c_2).$$

The key result:

Proposition 22. The Day convolution (def. 17) of two functors F and G is isomorphic to the left Kan extension of their external tensor product along $\otimes_C$:

$$F \underset{\text{Day}}{\otimes} G \cong \text{Lan}_{\underset{C}{\otimes}}(X \overline{\otimes} Y).$$

In other words, the Day convolution can be thought of as a left Kan extension.

Proof. This is immediate by spelling out definitions:

$$\begin{aligned} \text{Lan}_{\underset{C}{\otimes}}(F \overline{\otimes} G)(c) &\cong \int^{c_1, c_2 \in C} C\,(c_1 \underset{C}{\otimes} c_2, c) \underset{V}{\otimes} (F \overline{\otimes} G)(c_1, c_2) \\ &= \int^{c_1, c_2 \in C} C\,(c_1 \underset{C}{\otimes} c_2, c) \underset{V}{\otimes} F(c_1) \underset{V}{\otimes} G(c_2). \end{aligned}$$

■

Remark 23. Intuitively, we are approximating the external tensor product.

Corollary 24. There are natural isomorphisms

$$[C, V](F \underset{\text{Day}}{\otimes} G, H) \cong [C \times C, V](F \overline{\otimes} G, H \circ \underset{C}{\otimes}).$$

Remark 25. The point is, the structure associated to Day convolution can be rather abstract. The external tensor product and tensor product with respect to C can oftentimes be more tractable.

A particularly notable example of the usefulness of cor. 24 is in determining monoids with respect to the Day convolution. Certain functors (e.g. strong monoidal functors) arise as having the form on the right side of the natural isomorphism in cor. 24. Thus these functors *themselves* can be monoids with respect to the Day convolution. A notable example is *functors with smash product* arising arising in the construction of a symmetric monoidal smash (tensor) product of spectra via full subcategory inclusions from the category of pre-excisive functors (e.g. orthogonal, symmetric spectra). For details, see the *Introduction to Stable homotopy theory -- 1-2* article on the nLab.

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