C^*-algebras

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Unless otherwise stated, the field of scalars is assumed to be $\mathbb{C}$.

1 Banach algebras

Definition 1.1. A *Banach algebra* is a Banach space A paired with an associative, distributive multiplication such that

- (linearity) $\lambda(ab) = (\lambda a)b = a(\lambda b)$
- (continuity) $\|ab\| \leq \|a\| \cdot \|b\|$

for all $a, b \in A$ and all $\lambda \in \mathbb{C}$.

The following justifies the way we describe the second condition (though strictly speaking the second condition is stronger than continuity):

Proposition 1.2. Multiplication in a Banach algebra A is continuous (as a function $A \times A \to A$).

Proof. If $x_n \to x$ and $y_n \to y$ in A, then

$$\begin{aligned} \|x_n y_n - xy\| =& \|(x_n - x)y + (y_n - y)x_n\| \\ \leq& \|x_n - x\| \cdot \|y\| + \|y_n - y\| \cdot \|x_n\| \\ \to& 0. \end{aligned}$$

So $x_n y_n \to xy$. □

1.1 units

As one might expect, a unit $1_A \in A$ is an element such that $a 1_A = 1_A a = a$ for any $a \in A$. There is a connection between non-unital Banach algebras and unital Banach algebras, specifically in that we can embed any non-unital Banach algebra into a unital one as an ideal of codimension 1.

Corollary 1.3. If A has a unit, then the unit is unique.

Proof. Let $1_A, 1'_A$ be units. Then $1_A = 1_A 1'_A = 1'_A 1_A = 1'_A$. □

We would typically expect a unit to have norm 1. However, this is not always the case. The situation isn't far from this, however. We will show it is always true that $\|1_A\| \geq 1$. Furthermore, we can replace the norm on A with an equivalent one in which the (new) norm of 1_A is 1.

Proposition 1.4. If A is a unital Banach algebra, then $\|1_A\| \geq 1$.

Proof. Since $1_A = 1_A^2$, it follows $\|1_A\| \leq \|1_A\| \cdot \|1_A\|$. Such an inequality on real numbers only holds for numbers ≥ 1, hence the result. □

Lemma 1.5. If A is a unital Banach algebra, then there exists an equivalent norm n on A such that A is unital and $n(1_A) = 1$.

Proof. For $x \in A$, consider the left multiplication operator $L_x : y \mapsto xy$.

Claim 1.6. L_x is injective, bounded, and linear.

Proof of claim. For injectivity, suppose $L_x = L_{x'}$. Then in particular $x = x1_A = x'1_A = x'$. For linearity, note

$$L_{\lambda x+y}z = (\lambda x + y)z = \lambda xz + yz = (\lambda L_x + L_y)(z).$$

For boundedness, note

$$\|L_x y\| = \|xy\| \leq \|x\| \cdot \|y\|,$$

so L_x is bounded and in particular $\|L_x\| \leq \|x\|$. □

We now set

$$n(x) = \|L_x\|.$$

First let us show this norm is equivalent to $\|\cdot\|$. By the above, we already have $n(x) \leq \|x\|$. Conversely,

$$\begin{aligned} n(x) =& \|L_x\| = \sup\{\|L_x y\| : \|y\| \leq 1\} \\ =& \sup\{\|xy\| : \|y\| \leq 1\} \\ \geq& \|xy'\| \quad (\text{setting } y' = \frac{1_A}{\|1_A\|}) \\ =& \frac{\|x\|}{\|1_A\|}. \end{aligned}$$

In total,

$$\frac{\|x\|}{\|1_A\|} \leq n(x) \leq \|x\|$$

which proves equivalence.

Now let us show $n(-)$ makes A a Banach algebra. Completeness follows by equivalence, and for all $x, y \in A$ we have

$$n(xy) = \|L_{xy}\| = \|L_x L_y\|$$

$$\leq \|L_x\| \cdot \|L_y\| = n(x)n(y).$$

Thus A is a Banach algebra under $n(-)$.

ref bounded operators are banach algebra

Finally, we check that $n(1_A) = 1$:

$$\begin{aligned} n(1_A) =& \|L_{1_A}\| \\ =& \sup\{\|L_{1_A} y\| : \|y\| \leq 1\} = \sup\{\|y\| : \|y\| \leq 1\} \\ =& 1. \end{aligned}$$

This concludes the proof. □

Lemma 1.7. If A is a non-unital Banach algebra, then it can be embedded into a unital Banach algebra A_I as an ideal of codimension 1.

Remark 1.8. As our construction will show, $\|1_{A_I}\| = 1$. We don't mention it in the statement above because, by Lemma 1.5 we could always modify A_I so that this is the case.

Proof. Let $A_I = A \oplus \mathbb{C}$. Define multiplication on A_I as follows:

$$(x, \lambda)(y, \mu) = (xy + \mu x + \lambda y, \lambda\mu).$$

One checks this is associative and distributive. Also $(0, 1)$ is a unit:

$$(x, \lambda)(0, 1) = (x0 + x + \lambda 0, \lambda 1) = (x, \lambda).$$

We now define the norm on A_I as follows:

$$\|(x, \lambda)\| = \|x\| + |\lambda|.$$

Note $\|(0, 1)\| = 1$. Let us now verify that this norm makes A_I into a Banach space: suppose a sequence $\{(x_n, \lambda_n)\} \subset A_I$ is Cauchy. Then for any $\epsilon > 0$ there exists $N > 0$ such that for all $n, m \geq N$ we have $\|(x_n, \lambda_n) - (x_m, \lambda_m)\| < \epsilon$. But then

$$\|(x_n - x_m, \lambda_n - \lambda_m)\| = \|x_n - x_m\| + |\lambda_n - \lambda_m| < \epsilon,$$

so both $(x_n) \subset A$ and $(\lambda_n) \subset \mathbb{C}$ are Cauchy, hence convergent. Say $x_n \to x$ and $\lambda_n \to \lambda$. Then we can find $N' > 0$ such that

$$\|(x_n, \lambda_n) - (x, \lambda)\| = \|x_n - x\| + \|\lambda_n - \lambda\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This shows completeness.

Now let us check this norm makes A_I a Banach algebra:

$$\begin{aligned} \|(x, \lambda)(y, \mu)\| =& \|(xy + \mu x + \lambda y, \lambda\mu)\| = \|xy + \mu x + \lambda y\| + \|\lambda\mu\| \\ \leq& \|x\| \cdot \|y\| + |\mu| \cdot \|x\| + |\lambda| \cdot \|y\| + |\lambda| \cdot |\mu| \\ =& (\|x\| + |\lambda|)(\|y\| + |\mu|) \\ =& \|(x, \lambda)\| \cdot \|(y, \mu)\|. \end{aligned}$$

Hence A_I is a Banach algebra with unit.

All that is left is to identify A is a codimension 1 ideal in A_I. The mapping $x \mapsto (x, 0)$ is an isometric isomorphism between A is the set $M := \{(x, 0) : x \in A\} \subset A_I$. This set is codimension one, and an ideal since

$$\begin{aligned}(y, \lambda)(x, 0) =& (yx + \lambda x, 0) \in M,\\ (x, 0)(y, \lambda) =& (xy + \mu x, 0) \in M.\end{aligned}$$

This concludes the proof. □

1.2 invertible elements

In a unital Banach algebra A, the inverse of a, if it exists, is such that $a^{-1}a = aa^{-1} = 1_A$. There are two initial complications: the lack of a commutativity assumption means an element could satisfy one of the above conditions when multiplying on the left but not the right. The second is uniqueness.

We will write $\mathcal{G}(A)$ for the set (in fact, group) of invertible elements. Elements which are not invertible are called singular.

Proposition 1.9. Let $x \in A$. If l is a left inverse of x, i.e. $lx = 1_A$, and r is a right inverse of x, i.e. $xr = 1_A$, then $x \in \mathcal{G}(A)$ and, in particular, $l = r$.

Proof. By assumption $lx = xr = 1_A$. Then in particular $r = (lx)r = l(xr) = l$. □

Proposition 1.10. $\mathcal{G}(A) \subset A$ is open.

Proof. Recall that if $y \in A$ is such that $\|y\| < 1$, then

$$w = \sum_{k=0}^{\infty} y^k$$

exists by (*Banach spaces*, Proposition 5.1). Furthermore, w is the inverse of $1_A - y$ by the Neumann series (*Banach spaces*, Proposition 1.1). So if $x \in A$ is such that $\|1_A - x\| < 1$, then since $x = 1_A - (1_A - x)$ we get $x^{-1} = \sum_k (1_A - x)^k$.

We will show that for any $x_0 \in \mathcal{G}(A)$ we have $B(\frac{1}{\|x_0^{-1}\|}, x_0) \subset \mathcal{G}(A)$, which will complete the proof.

Now for any $x_0 \in \mathcal{G}(A)$, we know $x_0 x_0^{-1} x = x$. Also

$$\|1_A - x_0^{-1}x\| = \|x_0^{-1}(x_0 - x)\| \leq \|x_0\|^{-1} \cdot \|x_0 - x\|.$$

If $\|x - x_0\| < \|x_0^{-1}\|^{-1}$ (i.e. x is in the ball described above), then the above inequality becomes $\|1_A - x_0^{-1}x\| < 1$ so we can apply our previous remarks to get $x_0^{-1}x$ is invertible and in particular

$$(x_0^{-1}x)^{-1} = \sum_{k=0}^{\infty} (1_A - x_0^{-1}x)^k.$$

To show $x \in \mathcal{G}(A)$, we can write

$$(x_0^{-1}x)^{-1}(x_0^{-1}x) = \underbrace{\left(\sum_{k=0}^{\infty}(1_A - x_0^{-1}x)^k\right)(x_0^{-1}}_{x^{-1}}x) = 1_A.$$

This is a left inverse. A right inverse can be obtained analagously. By Proposition 1.9 this suffices to show $x \in \mathcal{G}(A)$. □

Corollary 1.11. The inversion function $\mathcal{G}(A) \to A$ is continuous.

Proof. Suppose $(x_n) \subset \mathcal{G}(A)$ converges to $x_0 \in \mathcal{G}(A)$. For large enough n, we get $\|x_n - x_0\| < \frac{1}{\|x_0^{-1}\|}$. Then by the above proof

$$\begin{aligned}\|x_n^{-1} - x_0^{-1}\| &= \left\|\sum_{k=0}^{\infty}((x_0^{-1}(x_0 - x_n))^k)x_0^{-1}\right\| \\ &\le \sum_{k=0}^{\infty}(\|x_0^{-1}\| \cdot \|x_0 - x_n\|)^k\|x_0^{-1}\|\end{aligned}$$

which tends to 0 since $\|x_0 - x_n\| \to 0$. In other words, $(x_n^{-1}) \to x_0^{-1}$. □

1.3 spectrum

Proposition 1.12. For $x \in X$, the resolvent set $\mathbb{C} - \sigma_A(x)$ is open.

Proof. We will show that for every $\lambda_0 \in \mathbb{C} - \sigma_A(x)$, the open ball

$$B := B(\lambda_0, \frac{1}{\|R_x(\lambda_0)\|})$$

is contained in $\mathbb{C} - \sigma_A(x)$. So let $\lambda \in B$. Then $\|(\lambda - \lambda_0)R_x(\lambda_0)\| < 1$, so *Banach spaces* Proposition 1.1 implies

$$1 - (\lambda - \lambda_0)R_x(\lambda_0)$$

is invertible. Since $\lambda_0 \notin \sigma(x)$, it is also true that $(x - \lambda_0 1)$ is invertible. Combining these facts, we can show that $x - \lambda 1$ is invertible:

$$x - \lambda 1 = x - \lambda_0 1 - (\lambda - \lambda_0)1 = (x - \lambda_0 1) \cdot (1 - (\lambda - \lambda_0)R_x(\lambda_0)).$$

Thus $\lambda \in \mathbb{C} - \sigma_A(x)$ and the result follows. □

Proposition 1.13. $\sigma_A(x) \subset \mathbb{C}$ is closed and bounded:

$$\sigma_A(x) \subset \{\lambda \in \mathbb{C} : |\lambda| \le \|x\|\}.$$

In particular, it is compact.

Proof. If $|\lambda| > \|x\|$, then $(x - \lambda 1_A) = -\lambda(1 - \frac{x}{\lambda})$ has an inverse provided by the Neumann series (*Banach spaces*, Proposition 1.1) since $\|\frac{x}{\lambda}\| < 1$. Hence $\sigma_A(x)$ is bounded. It is closed because its complement is open, by Proposition 1.12. □

Proposition 1.14. The resolvent function R_x is analytic (in the sense of *Banach spaces*, Definition 5.3).

Proof. For this claim to even make sense, the domain of R_x must be open. This is Proposition 1.12. We need to show that R_x can be defined as a power series which converges absolutely on an open disk centered at each $\lambda_0 \in \mathbb{C} - \sigma_A(x)$. By the proof of Proposition 1.12, we can take the disk of radius $\|R_x(\lambda_0)\|^{-1}$.

Let $\lambda \in \mathbb{C} - \sigma_A(x)$. In the proof of Proposition 1.12, we show that $x - \lambda 1)$ is invertible, and, for λ_0 in the disk described at the end of the previous paragraph,

$$x - \lambda 1 = (x - \lambda_0) \cdot (1 - (\lambda - \lambda_0) R_x(\lambda_0)).$$

Taking inverses of both sides,

$$R_x(\lambda) = R_x(\lambda_0) \cdot (1 - (\lambda - \lambda_0) R_x(\lambda_0))^{-1}.$$

Now we can expand $(1 - (\lambda - \lambda_0) R_x(\lambda_0))^{-1}$ as a Neumann series (Proposition 1.1) to get

$$R_x(\lambda) = R_x(\lambda_0) \sum_{n=0}^{\infty} R_x(\lambda_0)^n (\lambda - \lambda_0)^n = \sum_{n=0}^{\infty} R_x(\lambda_0)^{n+1} (\lambda - \lambda_0)^n,$$

which shows that R_x can be expressed as an appropriate power series. □

Theorem 1.15. $\sigma_A(x)$ is nonempty.

Proof. Suppose $\sigma_A(x)$ is empty. Then the resolvent is defined on all of $\mathbb{C}$. It is analytic (Proposition 1.14) and nonconstant. We claim it is also bounded. Note $\|x - 1_A\lambda\| \le \|x\| + |\lambda|$. So $\|(x - 1_A\lambda)^{-1}\|$ is bounded by $(\|x\| + |\lambda|)^{-1}$. In particular, it is finite away from infinity. It remains to show that it remains bounded as $|\lambda| \to \infty$. Without loss of generality, suppose $|\lambda| > \|x\|$. Then, by the Neumann series (*Banach spaces* Proposition 1.1) we get

$$\begin{aligned}(x - 1_A\lambda)^{-1} =& \frac{-1}{\lambda}\left(1_A - \frac{x}{\lambda}\right)^{-1} \\ =& \frac{1}{\lambda}\sum_{k=0}^{\infty}\left(\frac{x}{\lambda}\right)^k.\end{aligned}$$

Hence

$$\begin{aligned}\|(x - 1_A\lambda)^{-1}\| \le& \frac{1}{|\lambda|}\sum_{k=0}^{\infty}\left(\frac{\|x\|}{|\lambda|}\right)^k \\ =& \frac{1}{|\lambda|} \cdot \frac{1}{1 - \|x\|/|\lambda|}\end{aligned}$$

$$= \frac{1}{|\lambda| - \|x\|}.$$

This tends to 0 as $|\lambda| \to \infty$, hence R_x is bounded.

We have shown R_x is entire, bounded, and nonconstant. Thus by Louisville's theorem $R_x =$ 0. But then $(x - \lambda 1_A)^{-1} = 0$, contradicting the fact that $(x - \lambda 1_A)(x - \lambda 1_A)^{-1} = 1_A \neq 0$. ☐ ref

Proposition 1.16. Let A be a unital Banach algebra, and $a \in A$.

1. For any complex polynomial p, we have $\sigma_A(p(a)) = p(\sigma_A(a))$.
2. If $a \in \mathcal{G}(A)$, then $\sigma_A(a^{-1}) = \sigma_A(a)^{-1}$.

Proof. 1. Suppose $\deg(p) \geq 1$ (the other case is immediate). For any $\mu \in \mathbb{C}$, let $(\lambda_i)_1^n$ be the complex roots of the polynomial $p(-) - \mu$. In other words, for all $z \in \mathbb{C}$, we have

$$p(z) - \mu = \alpha(z - \lambda_1) \cdots (z - \lambda_n)$$

for some $\alpha \in \mathbb{C}$. Then

$$p(a) - \mu 1_A = \alpha(a - \lambda_1 1_A) \cdots (a - \lambda_n 1_A).$$

Claim 1.17. If $(a_i)_1^n \subset A$ are mutually commuting, then the product $a_1 \cdots a_n$ is invertible if and only if each a_i is invertible.

Proof of claim. In one direction, if each a_i are invertible then $a_1 \cdots a_n$ is invertible, regardless even of the mutually commuting assumption.

Conversely, if the product $a_1 \cdots a_n$ is invertible, then we can write down an inverse for each a_i. For example,

$$\begin{aligned} a_2(a_1 a_3 \cdots a_n)(a_1 \cdots a_n)^{-1} &= (a_1 \cdots a_n)^{-1}(a_1 \cdots a_n) \\ &= 1_A \\ &= (a_1 \cdots a_n)^{-1}(a_1 a_3 \cdots a_n) a_2. \end{aligned}$$

This shows the claim. ☐

First we will show

$$\sigma_A(p(a)) \subset p(\sigma_A(a)).$$

Suppose $\mu \in \sigma_A(p(a))$. Then by definition $p(a) - \mu 1_A$ is singular, so by the claim there exists some i such that $a - \lambda_i 1_A$ is singular. This would mean $\lambda_i \in \sigma_A(a)$. But

$$p(\lambda_i) - \mu = \alpha(\lambda_i - \lambda_1) \cdots (\lambda_i - \lambda_n) = 0,$$

so $p(\lambda_i) = \mu$. Hence $\mu \in p(\sigma_A(a))$ as desired.

Now we will show

$$p(\sigma_A(a)) \subset \sigma_A(p(a)).$$

Suppose $\lambda \in \sigma_A(a)$, and write $\mu := p(\lambda)$ (so that $\mu \in p(\sigma_A(a))$). Then again

$$p(z) - \mu = \alpha(z - \lambda_1)\cdots(z - \lambda_n),$$

so if $z = \lambda$ then $p(z) - \mu = 0$, i.e. $p(z) = \mu$. By the above remarks we would get that $z = \lambda = \lambda_i$ for some i. Hence $(a - \lambda_i 1_A)$ is singular, since $(a - \lambda 1_A)$ is by assumption. By the claim, we get that

$$p(a) - \mu 1_A = \alpha(a - \lambda_1 1_A)\cdots(a - \lambda_n 1_A)$$

is singular. This shows $\mu := p(\lambda) \in \sigma_A(p(a))$, which is what we wanted.

2. If $a \in \mathcal{G}(A)$, then by definition $0 \neq \sigma_A(a)$. For any $\lambda \in \mathbb{C}$ which is nonzero, we get

$$a - \lambda 1_A = a(1_A - \lambda a^{-1}) = a\lambda(\lambda^{-1}1_A - a^{-1}).$$

Since $a\lambda$ is invertible, it follows that $a - \lambda 1_A$ is invertible if and only if $\lambda^{-1}1_A - a^{-1}$ is invertible, which is only the case if its negative $a^{-1} - \lambda^{-1}1_A$ is invertible. This is precisely the statement in the proposition.

□

Corollary 1.18. $\sigma_A(-a) = -\sigma(a)$.

Proposition 1.19. Let R be a unital ring. The element $1 - yx$ is invertible if and only if $1 - xy$ is invertible. In particular,

$$(1 - yx)^{-1} = 1 - y(1 - xy)^{-1}x.$$

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Proposition 1.20. Let A be a unital Banach algebra. Then

$$\sigma_A(xy) \cup \{0\} = \sigma_A(yx) \cup \{0\}.$$

Proof. Let $\lambda \in \mathbb{C}$ be nonzero. Then

$$\begin{aligned} \lambda \in \sigma_A(xy) &\Longleftrightarrow \lambda 1_A - xy \text{ invertible} \\ &\Longleftrightarrow \lambda\left(1_A - \frac{xy}{\lambda}\right) \text{ invertible.} \end{aligned}$$

But by Propostion 1.19, this is only possible if

$$\lambda\left(1_A - \frac{yx}{\lambda}\right) = \lambda 1_A - yx$$

is invertible, i.e. if and only if $\lambda \in \sigma_A(yx)$. □

Proposition 1.21. Let A be a unital Banach algebra, and $a, b \in A$. Then $ab - ba \neq 1_A$.

Proof. Suppose $ab - ba = 1_A$. One one hand, Proposition 1.20 says

$$\sigma_A(ab) \cup \{0\} = \sigma_A(ba) \cup \{0\}.$$

But on the other hand

$$\sigma_A(ab) = \sigma_A(1_A + ba) = 1 + \sigma_A(ba)$$

by . This suggests ref

$$\{1 + \sigma_A(ba)\} \cup \{0\} = \sigma_A(ba) \cup \{0\},$$

which we will now show is a contradiction.

First suppose $\alpha \in \sigma_A(ba)$ is such that $\mathrm{Re}(\alpha) \geq 0$. Then $1 + \alpha \neq 0$, so $(1 + \alpha) \in \{1 + \sigma_A(ba)\} \setminus \{0\}$, implying by the above equality that $1 + \alpha \in \sigma_A(ba)$. Inductively we show $n + \alpha \in \sigma_A(ba)$ for all n, contradicting the boundedness of $\sigma_A(ba)$ (Proposition 1.13).

Now suppose $\mathrm{Re}(\alpha) < 0$. Observe $-\alpha \in \sigma_A(-ba)$, and $\mathrm{Re}(-\alpha) > 0$, so we reach a contradiction by the above situation. ☐ correct?

Theorem 1.22 (Gelfand-Mazur)**.** Let A be a unital Banach algebra. If every nonzero element is invertible, then $A = \mathbb{C}$.

Proof. Let $x \in A$. Then $\sigma_A(x) \neq \emptyset$ so there exists $\lambda \in \mathbb{C}$ such that $x - \lambda 1_A \notin \mathcal{G}(A)$. By assumption this means $x - \lambda 1_A = 0$, so $x = \lambda 1_A$. ☐

Theorem 1.23 (Gelfand's formula)**.**

$$\rho(x) = \limsup_n \|x^n\|^{1/n}.$$

Proof. First we will show $\rho(x) \leq \limsup_n \|x^n\|^{1/n}$. Indeed, if $|\lambda| > \limsup_n \|x^n\|^{1/n}$, then $\limsup_n \|\lambda^{-n} x^n\| < 1$ so, by *Banach spaces* Proposition 1.3, $(\lambda 1 - x)^{-1}$ exists. Since this is true for all λ such that $|\lambda| > \limsup_n \|x^n\|^{1/n}$, it must be that $\limsup_n \|x^n\|^{1/n} \geq \rho(x)$ (else there would be a λ for which $(\lambda 1 - x)$ is not invertible, contradicting what we have just shown).

Conversely, suppose λ is such that $|\lambda| > \rho(x)$.

Claim 1.24.

$$\lim_{n\to\infty} \phi\left(\frac{x^n}{\lambda^{n+1}}\right) = 0.$$

Proof. By Proposition 1.14, the resolvent is analytic on the set of λ such that $|\lambda| > \rho(x)$. By the uniqueness of the Laurent expansion () and *Banach spaces* Corollary 1.3, we must have ref

$$(\lambda 1 - x)^{-1} = \sum_{k=0}^{\infty} \frac{x^k}{\lambda^{k+1}}.$$

Let $\phi \in X^*$. Since $(\lambda 1 - x)^{-1}$ exists, so must $\phi((\lambda 1 - x)^{-1})$, i.e.

$$\phi\left(\sum_{k=0}^{\infty} \frac{x^k}{\lambda^{k+1}}\right) = \sum_{k=0}^{\infty} \phi\left(\frac{x^k}{\lambda^{k+1}}\right)$$

is convergent, where we have used the continuity and linearity of ϕ. The result follows. □

Define

$$T_n : X^* \to \mathbb{C}$$
$$\phi \mapsto \phi\left(\frac{x^n}{\lambda^{n+1}}\right)$$

for all $n \in \mathbb{N}$. Since each T_n is an evaluation map, it is continuous and linear. We also have that X^* (and $\mathbb{C}$) are Banach. Claim 1.24 implies

$$\sup_{T \in \{T_n\}} \|T(\phi)\| < \infty$$

for all $\phi \in X^*$. Hence by the uniform boundedness principle (*Banach spaces* Theorem 3.9)

$$\sup_{T \in \{T_n\}} \|T\| < \infty.$$

Let's unwrap what this gives us. By definition

$$\|T\| = \sup_{\|\phi\|=1} \left\|\phi\left(\frac{x^n}{\lambda^{n+1}}\right)\right\| = \frac{1}{\lambda^{n+1}} \sup_{\|\phi\|=1} \|\phi(x^n)\|.$$

Also by definition,

$$\|\phi\| = \sup_{x \neq 0} \frac{|\phi(x)|}{\|x\|},$$

so $\|\phi\| = 1$ means, in particular, $\|\phi(x^n)\| \leq \|x^n\|$. By , there actually exists ϕ such that $\|\phi(x^n)\| = \|x^n\|$ and $\|\phi\| = 1$. Hence

ref norming functional

$$\sup_{\|\phi\|=1} \|\phi(x^n)\| = x^n.$$

Pulling it all together, we have

$$\|T\| = \left\|\frac{x^n}{\lambda^{n+1}}\right\|.$$

Since this is true for all n, the sequence (x^n/λ^{n+1}) is bounded, i.e. there exists $\alpha > 0$ such that $\|x^n/\lambda^{n+1}\| < \alpha$ for all n. Hence $\|x^n\|^{1/n} < \alpha^{1/n}|\lambda|^{n+1/n}$. Then

$$\limsup_n \|x^n\|^{1/n} \leq \limsup_n \alpha^{1/n}|\lambda|^{n+1/n} = |\lambda|.$$

Since we chose $|\lambda| > \rho(x)$ arbitrarily, we have shown

$$\limsup_n \|x^n\|^{1/n} \leq \rho(x),$$

completing the proof. □

1.4 constructing Banach algebras

Proposition 1.25. Let A be a Banach algebra, and V a closed 2-sided ideal. Then A/V is a Banach algebra. If moreover A is unital and V is proper, then A/V is unital and $1_{A/V} \leq \|1_A\|$.

Proof. By *Banach spaces*, Proposition 2.5, we know A/V is a Banach space.

We first claim A/V is an algebra with respect to the multiplication $[x] \cdot [y] = [xy]$. Indeed, if $x' \sim x$ and $y' \sim y$, then

$$xy - x'y' = (x - x')y + x'(y - y') \in V$$

since V is a 2-sided ideal.

For the Banach inequality, we calculate

$$\begin{aligned}
\|[x] \cdot [y]\| =& \|[xy]\| = \inf_{v \in V} \|xy + v\| \\
\leq& \inf_{v,w \in V} \|xy + \underbrace{xw + vy + vw}_{\in V}\| \\
=& \inf_{v,w \in V} \|(x + v)(y + w)\| \\
\leq& \inf_{v,w \in V} \|x + v\| \cdot \|y + w\| \\
=& \|[x]\| \cdot \|[y]\|.
\end{aligned}$$

Thus A/V is a Banach algebra.

Now suppose A is unital and V is proper. Immediately we have $[1_A] \cdot [b] = [b]$, so $[1_A]$ is a unit. Furthermore,

$$\|[1_A]\| = \inf_{v \in V} \|1_A + v\| \leq \|1_A\|$$

by taking $v = 0$. □

1.5 Gelfand theory

Proposition 1.26. Every maximal ideal in a unital Banach algebra is closed.

Proof. Let $J \subset A$ be a maximal ideal. Then J cannot contain any invertible element (otherwise $J = A$). Hence $J \subset A \setminus \mathcal{G}(A)$. By Proposition 1.10, $\mathcal{G}(A) \subset A$ is open, so $A \setminus \mathcal{G}(A)$ is closed, hence

$$J \subset \overline{J} \subset A \setminus \mathcal{G}(A).$$

In particular, $\overline{J} \neq A$. Since J is maximal and $\overline{J}$ is also an ideal, it follows that $J = \overline{J}$. □

Definition 1.27. Let A be a Banach algebra. A nonzero homomorphism $A \to \mathbb{C}$ is called a *character* of A. The set of all characters on A is called the *spectrum*, and denoted $\mathrm{Sp}(A)$.

Corollary 1.28. For a unital Banach algebra A and any $\ell \in \mathrm{Sp}(A)$, we have $\ell(1_A) = 1$.

Proof. We calculate

$$\phi(a) = \phi(a1_A) = \phi(a)\phi(1_A),$$

so $\phi(1_A) = 1$. □

Corollary 1.29. Let A be a unital Banach algebra, and $\ell : A \to \mathbb{C}$ a character. For any $a \in A$, we have $\ell(a) \in \sigma_A(a)$.

Proof. First calculate

$$\ell(a - \ell(a)1_A) = \ell(a) - \ell(a) = 0.$$

Now if $a - \ell(a)1_A$ has an inverse b, then

$$\ell((a - \ell(a)1_A)b) = \ell(a - \ell(a)1_A)\ell(b) = 0 \cdot b = 0,$$

contradicting Corollary 1.28. □

Proposition 1.30. Every character ϕ on a Banach algebra is continuous. In particular, $\|\phi\|_\infty \leq 1$.

Proof. Let A be a Banach algebra, and ϕ a character. First suppose A is unital. For any $a \in A$ such that $\phi(a) \neq 0$, Corollary 1.29 tells us that $\phi(a) \in \sigma_A(a)$. Proposition 1.13 tells us that $|\phi(a)| \leq \|a\|$. The inequality still holds if $\phi(a) = 0$. Hence ϕ is continuous. Now suppose A is non-unital. Consider its unitization $A_I = A \oplus \mathbb{C}$, and the map

$$\begin{aligned} \phi' : A_I &\longrightarrow \mathbb{C} \\ (a, \lambda) &\mapsto \phi(a) + \lambda. \end{aligned}$$

One checks this is a homomorphism. It is continuous by the previous paragraph, hence its restriction to A (which is ϕ) is continuous as well. □

Theorem 1.31 (character correspondence)**.** Let A be a commutative unital Banach algebra. There is a canonical bijection

$$\begin{array}{ccc} \left\{ \begin{array}{c} \text{characters} \\ \text{of } A \end{array} \right\} & \longleftrightarrow & \left\{ \begin{array}{c} \text{maximal ideals} \\ \text{of } A \end{array} \right\} \\ \ell & \mapsto & \ker(\ell) \end{array}$$

Proof. The idea is that kernels are maximal ideals, and conversely every element outside a maximal ideal J is invertible, hence by the Gelfand-Mazur theorem $A/J \cong \mathbb{C}$. Then composition with the projection $A \to A/J$ uniquely determines a character. In detail:

Let $\ell : A \to \mathbb{C}$ be a character, and write $J = \ker(\ell)$. Since ℓ is nonzero by definition, $J \neq A$, so there exists $a \notin J$. Then any $b \in A$ can be written as

$$b = a\frac{\ell(b)}{\ell(a)} + \left(b - a\frac{\ell(b)}{\ell(a)}\right).$$

Note $b - a\frac{\ell(b)}{\ell(a)} \in \ker(\ell) = J$, since

$$\ell\left(b - a\frac{\ell(b)}{\ell(a)}\right) = \ell(b) - \ell(a)\frac{\ell(b)}{\ell(a)} = 0.$$

Since $b \in A$ is arbitrary, this shows $A = \mathbb{C}a + J$. Since J is codimension 1 it is maximal . better proof

Conversely, suppose J is maximal. By Proposition 1.26, J is closed, hence by Proposition 1.25 we have that A/J is a Banach algebra.

Claim 1.32. Every nonzero $[a] \in A/J$ is invertible. separate

Proof of claim. Suppose $[a]$ is not invertible. Then $J + aA$ is a proper ideal of A containing J, contradicting the maximality of J. □

By the claim, every (nonzero) element in A/J is invertible. Hence by the Gelfand-Mazur theorem (Theorem 1.22), there is an isomorphism $\phi : A/J \to \mathbb{C}$. Now let $\pi : A \to A/J$ be the canonical projection. Then $\phi \circ \pi : A \to A/J \to \mathbb{C}$ is a homomorphism with kernel J:

$$\begin{aligned}\phi \circ \pi(ab) =& \phi(\pi(ab)) = \phi([ab]) \\ =& \phi([a][b]) = \phi([a])\phi([b]) \\ =& (\phi \circ \pi(a))(\phi \circ \pi(b)),\end{aligned}$$

where $\phi \circ \pi(a) = 0$ if and only if $\pi(a) = 0$, if and only if $a \in J$.

This correspondence is one-to-one since ℓ is uniquely determined by its kernel: if $\ker(\ell) = \ker(\ell')$, then for all $a \in A$ we have $a - \ell(a)1_A \in \ker(\ell) = \ker(\ell')$, so $\ell'(a) = \ell(a)$ since $\ell'(1_A) = 1$. □

Proposition 1.33. Any commutative unital Banach algebra posseses at least one character.

Proof. If every element is invertible, then $A \cong \mathbb{C}$ by the Gelfand-Mazur theorem (Theorem 1.22), and this isomorphism is itself a character.

Otherwise, there exists a noninvertible $x \in A$. Then $xA \subset A$ is a proper ideal, hence contained in some maximal ideal J . By the character correspondence (Theorem 1.31), J is the kernel of a character on A. □ ref

Definition 1.34. The set of characters of a commutative unital Banach algebra A is called the *spectrum* of A, and is denoted $\mathrm{Sp}(A)$.

The natural topology in which to consider $\mathrm{Sp}(A)$ is the weak*-topology (*Banach spaces*, Definition 2.1):

Proposition 1.35. $\mathrm{Sp}(A) \subset A$ is compact in the weak*-topology. In particular, it is a (weak*-)closed subset of the unit ball in A^*.

Proof. By Proposition 1.30, we know that any character is bounded with norm ≤ 1, hence $\mathrm{Sp}(A)$ is contained in the unit ball in A^*.

Now suppose $(\ell_\alpha) \subset \mathrm{Sp}(A)$ is a net converging to $\phi \in A^*$. Then by definition of the weak*-topology, $\ell_\alpha(x) \to \phi(x)$ for all $x \in A$. But then, for all $x, y \in A$,

$$\phi(xy) = \lim \ell_\alpha(xy) = \lim \ell_\alpha(x)\ell_\alpha(y) = \phi(x)\phi(y).$$

Hence $\phi \in \mathrm{Sp}(A)$, so $\mathrm{Sp}(A)$ is a closed subset of the unit ball in A^*. Since the unit ball is compact in the weak*-topology by the Banach-Alaoglu theorem (*Banach spaces*, Theorem **??**), it follows that $\mathrm{Sp}(A)$ is compact. □

Definition 1.36. Let A be a commutative unital Banach algebra. For each $x \in A$, we can consider the "evaluation" map $\hat{x} : \mathrm{Sp}(A) \to \mathbb{C}$, i.e. the map whose value on $\ell \in \mathrm{Sp}(A)$ is

$$\hat{x}(\ell) = \ell(x).$$

Varying over all x, we obtain a map

$$\begin{aligned} X &\xrightarrow{\widehat{(-)}} C(\mathrm{Sp}(A)) \\ x &\mapsto (\ell \overset{\hat{x}}{\mapsto} \ell(x)). \end{aligned}$$

called the *Gelfand transform*.

Theorem 1.37. Let A be a commutative unital Banach algebra.

1. For any $x \in A$,
$$\mathrm{Im}(\hat{x}) = \sigma_A(x).$$
If moreover x generates A, i.e. the polynomials in x are dense in A, then the map $\hat{x}$ is a homeomorphism.

2. The Gelfand transform $\widehat{(-)} : A \to C(\mathrm{Sp}(A))$ is a homomorphism (in particular, each $\hat{x}$ is continuous) and
$$\|\hat{x}\|_\infty \le \|x\|$$
for all $x \in A$.

Proof.

1. Corollary 1.29 tells us that $\mathrm{Im}(\hat{x}) \subset \sigma_A(x)$. Conversely, let $x \in \sigma_A(x)$. So $x - \lambda 1_A$ is not invertible, so it belongs to some maximal ideal J. By the character correspondence (Theorem 1.31), there exists a (unique) $\ell \in \mathrm{Sp}(A)$ such that $\ker(\ell) = J$. Then $x - \lambda 1_A \in J$ miplies $\ell(x - \lambda 1_A) = 0$, which implies $\ell(x) = \lambda$. Hence $\lambda \in \mathrm{Im}(\hat{x})$, which shows $\sigma_A(x) \subset \mathrm{Im}(\hat{x})$.

2. To see that the Gelfand transform is a homomorphism, compute
$$\widehat{xy}(\ell) = \ell(xy) = \ell(x)\ell(y) = \hat{x}(\ell)\hat{y}(\ell).$$

To see that $\hat{x}$ is continuous, let $(\ell_\alpha) \subset \mathrm{Sp}(A)$ is a net converging to ℓ. Then by definition of the weak*-topology,

$$\hat{x}(\ell_\alpha) = \ell_\alpha(x) \to \ell(x) = \hat{x}(\ell).$$

This shows $\hat{x}$ is continuous.

The inequality is implied by the first part.

□

Corollary 1.38. Let A be a commutative unital Banach algebra generated by $a \in A$. Then $\hat{a} : \mathrm{Sp}(A) \to \sigma_A(a) \subset \mathbb{C}$ is a homeomorphism.

Proof. By Propositions 1.35 and 1.13, both $\mathrm{Sp}(A)$ and $\sigma_A(a)$ are compact Hausdorff. By Theorem 1.37, $\hat{a}$ is continuous and surjective in this case. By , it suffices to show $\hat{a}$ is injective. So suppose $\hat{a}(\ell_1) = \hat{a}(\ell_2)$, i.e. $\ell_1(a) = \ell_2(a)$. Then for any $c_0, c_1, \ldots, c_N \subset \mathbb{C}$, so

$$\ell_1\left(\sum_{n=0}^{N} c_n a^n\right) = \ell_2\left(\sum_{n=0}^{N} c_n a^n\right).$$

Since ℓ_1, ℓ_2 are continuous and a generates A, we have $\ell_1 = \ell_2$ and are done. □

ref that cts bij between compact hausdorff is homeo

2 C^*-algebras

Definition 2.1. Consider a Banach algebra A with an *involution* $a \mapsto a^*$ such that

1. (conjugate linear) $(\lambda a)^* = \overline{\lambda} a^*$.
2. $a^{**} = a$
3. $(ab)^* = b^* a^*$
4. (continuity) $\|a^*\| = \|a\|$
5. (C^*-property) $\|a^* a\| = \|a\|^2$.

If A satisfies properties 1-4, it is called a *Banach $*$-algebra*. If A satisfies all properties 1-5, it is called a *C^*-algebra*.

Definition 2.2. An element a in a C^*-algebra is called:

- *self-adjoint* if $x^* = x$.
- a *projection* if it is self-adjoint and $x^2 = x$.
- *normal* if $a^* a = a a^*$.
- *unitary* if it is normal and $a a^* = a^* a = 1_A$.

Corollary 2.3 (real-imaginary decomposition)**.** Given a C^*-algebra A, we can decompose any $x \in A$ as follows:

$$x = \frac{1}{2}(x + x^*) + i\frac{1}{2i}(x - x^*).$$

This is the unique decomposition of x as $x = h + ik$ where h, k are self-adjoint.

Proof. If we can write $x = h + ik$, then $x^* = h - ik$. Solving for h, k yields the desired result. □

Definition 2.4. Given a unital C^*-algebra A and an element $x \in A$, write $A(x)$ for the unital C^*-algebra generated by x, i.e. the closure in A of the $*$-algebra of complex polynomials in x, x^*, and 1_A.

Corollary 2.5. *$A(x)$ is commutative if and only if x is normal.*

Proposition 2.6. *Let A be a unital C^*-algebra, and $h \in A$ self-adjoint. Then $\sigma_A(h) \subset \mathbb{R}$.*

Proof. The idea is the following. $A(h)$ is commutative unital. Consider the exponential $u_t = e^{ith} \in A(h)$. One shows $\|u_t\| = 1$ for all $t \in \mathbb{R}$. Since for any $\ell \in \mathrm{Sp}(A(h))$ we have $\|\ell\| \leq 1$, we get $|\ell(u_t)| \leq \|u_t\| = 1$. We then use the continuity of ℓ to show $\ell(u_t) = u_{\ell(t)}$. So we end up with an inequality $|\exp(it\ell(h))| \leq 1$ for all $t \in \mathbb{R}$. This will imply that $\ell(h) \in \mathbb{R}$, and hence we conclude that $\hat{h}$ is real-valued. We now use the fact that $A(h)$ is commutative unital to know that $\sigma_{A(h)}(h) = \mathrm{im}(\hat{h})$. But $\sigma_A(h) \subset \sigma_{A(h)}(h)$ and we're done. In detail:

Suppose $h \in A$ is self-adjoint. By Corollary 2.5, $A(h)$ is a commutative, unital C^*-algebra. For $t \in \mathbb{R}$, write

$$u_t \coloneqq e^{ith} \coloneqq \sum_{n=0}^{\infty} \frac{(it)^n}{n!} h^n,$$

which is well-defined by . By the continuity of the involution,

add and ref

$$u_t^* = \lim_{n\to\infty} \left(\sum_{k=0}^{n} \frac{(it)^k}{k!} h^k \right)^* = \lim_{n\to\infty} \sum_{k=0}^{\infty} \frac{(-it)^k)}{k!} h^k$$
$$= u_{-t}$$

So

$$u_t^* u_t = u_{-t} u_t = u_0 = 1_A,$$

and $1 = \|u_t^* u_t\| = \|u_t\|^2$, implying $\|u_t\| = 1$ for all $t \in \mathbb{R}$.

Proposition 1.30 tells us that $\|\ell\| \leq 1$, so $|\ell(u_t) \leq \|u_t\| = 1$. Now let $\ell \in \mathrm{Sp}(A(h))$. Also by Proposition 1.30, ℓ is continuous, and so

$$\ell(u_t) = \ell \left(\sum_{n=0}^{\infty} \frac{(it)^n}{n!} h^n \right) = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} \ell(h)^n = e^{it\ell(h)}.$$

Combining these results tells us that $|e^{it\ell(h)}| \leq 1$ for all $t \in \mathbb{R}$. But this implies $\ell(h) \in \mathbb{R}$ (since, for example, $|e^z| \leq 1$ implies $\mathrm{Re}(z) \leq 0$, and our statement is true for all $t \in \mathbb{R}$ hence $\mathrm{Re}(\ell(h)) = 0$).

So $\hat{h}$ is real-valued. By Theorem 1.37, $\sigma_{A(h)}(h) = \mathrm{Im}(\hat{h})$, so $\sigma_{A(h)}(h) \subset \mathbb{R}$. But $A(h) \subset A$, so we also have $\sigma_A(h) \subset \sigma_{A(h)}(h)$, so $\sigma_A(h) \subset \mathbb{R}$ as desired. $\square$

Theorem 2.7 (Gelfand-Naimark)**.** *Let A be a commutative unital Banach $*$-algebra. The Gelfand transformation*

$$\widehat{(-)} : A \longrightarrow C(\mathrm{Sp}(A))$$

is an isometric $*$-isomorphism if and only if A is a C^*-algebra.

Proof. The idea is the following. For the forward direction, recall that $C(\mathrm{Sp}(A))$ is a C^*-algebra by . So if A is isometrically $*$-isomorphic to it, then so is A. Conversely, you can use the real-imaginary decomposition of $x \in A$ to show $\widehat{(-)}$ is a $*$-homomorphism. To see it is isometric (hence injective), we start with self-adjoint $h \in A$ and see ref ex

$$\|\hat{h}\|_\infty = \rho(h) = \limsup_n \|h^{2^n}\|^{1/2^n} = \|h\|.$$

The extension to general $x \in A$ is obtained by considering the self-adjoint element xx^*. To see it is surjective, we see first note that $\mathrm{im}(\widehat{(-)})$ is closed, then that it seperates points, hence is dense in, and in fact equal to, $C(\mathrm{Sp}(A))$. In detail:

We will just consider the converse direction. Suppose A is a C^*-algebra. By Proposition 2.6, for any self-adjoint $h \in A$ we have $\sigma_A(h) \subset \mathbb{R}$, i.e. $\hat{h}$ is real valued (e.g. by Theorem 1.37). By Corollary 2.3 we can write

$$x = \frac{1}{2}(x + x^*) + i\frac{1}{2i}(x - x^*)$$

for any $x \in A$. Then

$$\begin{aligned}\widehat{x^*}(\ell) =& \ell(x^*) = \ell\left(\frac{x+x^*}{2} - i\frac{(x-x^*)}{2i}\right)\\ =& \ell\left(\frac{x+x^*}{2}\right) - i\ell\left(\frac{x-x^*}{2i}\right)\\ =& \left(\ell\left(\frac{x+x^*}{2}\right) + i\ell\left(\frac{x+x^*}{2i}\right)\right)^-\\ =& \overline{\ell(x)} = \overline{\hat{x}(\ell)}.\end{aligned}$$

This shows $\widehat{(-)}$ is a $*$-homomorphism.

Now we will show $\widehat{(-)}$ is isometric (which will also show it is injective). First we will show it is isometric on self-adjoint elements. Let $h \in A$ be self-adjoint. By the C^*-property, $\|h\|^{2^n} = \|h^{2^n}\|$. Thus

$$\|\hat{h}\|_\infty = \rho(h) = \limsup_n \|h^{2^n}\|^{1/2^n} = \|h\|,$$

where we have used and Theorem 1.23. This shows $\widehat{(-)}$ is an isometric on self-adjoint elements. In the general case, let $x \in A$. Then ref

$$\begin{aligned}\|\hat{x}\|_\infty^2 =& \|\overline{\hat{x}}\hat{x}\|_\infty = \|\widehat{x^*x}\|_\infty\\ =& \|x^*x\| = \|x\|^2,\end{aligned}$$

where we have used the fact that x^*x is self-adjoint and the C^*-algebra property. So $\widehat{(-)}$ is isometric on all of A.

It remains to show $\widehat{(-)}$ is surjective. Since A is complete and $\widehat{(-)}$ is an isomorphism, $\mathrm{Im}(\widehat{(-)})$ is closed in $C(\mathrm{Sp}(A))$. In fact, it is a closed $*$-subalgebra with unit, since $\widehat{(-)}$ is a ref

$*$-homomorphism. We claim $\mathrm{im}(\widehat{(-)})$ seperates points of $\mathrm{Sp}(A)$. Indeed, for any $\ell_1 \neq \ell_2$ in $\mathrm{Sp}(A)$, by definition there exists $x \in A$ such that $\ell_1(x) = \ell_2(x)$, i.e. $\hat{x}(\ell_1) \neq \hat{x}(\ell_2)$. Then, by the Stone-Weierstrass theorem (), $\mathrm{im}(\widehat{(-)}) \subset C(\mathrm{Sp}(A))$ is dense. Since it is also closed, we have equality and we are done. ∎

Theorem 2.8. Let A be a unital C^*-algebra, and $x \in A$ an invertible element. Then x^{-1} belongs to the C^*-subalgebra of A generated by $1_A, x, x^*$ (i.e. the closure of A in the set of complex polynomials in $1_A, x, x^*$).

Proof. The idea is the following. We will first consider a self-adjoint element x. In this case, we can actually show that x^{-1} is in the algebra generated by x. To see this, we let $\mathcal{A}$ be the algebra generated by x, and $\mathcal{B}$ be the algebra generated by x^{-1}. Since x and x^{-1} commute, $\mathcal{B}$ is commutative, so it is isometrically $*$-isomorphic to $C(\mathrm{Sp}(\mathcal{B}))$ by the Gelfand-Naimark theorem. Now one shows the image of $\mathcal{A}$ under the Gelfand transform, $\hat{\mathcal{A}}$, separates the points of $\mathrm{Sp}(\mathcal{B})$. So by Stone-Weierstrass we conclude $\hat{\mathcal{A}} = \hat{\mathcal{B}}$. The Gelfand-Naimark theorem in the other direction then gives us that $\mathcal{A} = \mathcal{B}$, and in particular $x^{-1} \in \mathcal{A}$. For the general case, we do a similar thing, observing that for the self-adjoint element x^*x a certain element appears in the C^*-algebra generated by $1_A, x^*x$ which will also appear in the algebra generated by $1_A, x, x^*$ and will multiply with x^* to yield x^{-1}.

Suppose first that $x = x^*$. Let $\mathcal{A}$ be the unital C^*-algebra generated by x, and $\mathcal{B}$ the unital C^*-algebra generated by x, x^{-1}. So $\mathcal{A} \subset \mathcal{B} \subset A$. Since x and x^{-1} commute, $\mathcal{B}$ is commutative. Thus by Theorem 2.7, the Gelfand transform $\widehat{(-)} : \mathcal{B} \to \hat{\mathcal{B}} := C(\mathrm{Sp}(B))$ is an isometric $*$-isomorphism. Since $\mathcal{A}$ is a C^*-subalgebra of $\mathcal{B}$, it follows () that $\hat{A}$ (the image of A under the Gelfand transform) is a C^*-subalgebra of $\hat{B}$.

We claim $\hat{A}$ separates points of $\mathrm{Sp}(B)$. Let $\ell_1, \ell_2 \in \mathrm{Sp}(B)$, and suppose $\ell_1(x) = \ell_2(x)$. Then for any $\ell \in \mathrm{Sp}(B)$,

$$\ell(xx^{-1}) = \ell(x)\ell(x^{-1}) = \ell(1_A) = 1_A$$

and

$$\ell_1(x^{-1}) = \ell(x)^{-1} = \ell_2(x)^{-1} = \ell_2(x^{-1}).$$

Since B is generated by x, x^{-1} it follows that $\ell_1 = \ell_2$. We have shown that if $\ell_1 \neq \ell_2$, then $\ell_1(x) \neq \ell_2(x)$, i.e. $\hat{x}(\ell_1) \neq \hat{x}(\ell_2)$. So $\hat{\mathcal{A}}$ separates points of $\mathrm{Sp}(B)$. So $\hat{\mathcal{A}} \subset \hat{\mathcal{B}}$ is dense, and since it is also closed, they are equal. Theorem 2.7 again now imiplies $A = B$. In particular, $x^{-1} \in \mathcal{A}$.

For the general case, consider an invertible element $x \in A$. Then x^*x is invertible with inverse $x^{-1}(x^{-1})^*$. But x^*x is self-adjoint, hence by the above $x^{-1}(x^{-1})^*$ is in the C^*-algebra generated by 1_A and x^*x, which itself is in the C^*-algebra generated by $1_A, x, x^*$. But then

$$x^{-1}(x^{-1})^*x^* = (x^*x)^{-1}x^* = x^{-1}$$

is also in that algebra, and we are done. ∎

Corollary 2.9 (spectral permanence)**.** Let $A \subset B$ be unital C^*-algebras with the same unit, and let $x \in A$. Then $\sigma_A(x) = \sigma_B(x)$.

Proof. $A \subset B$ already implies $\sigma_B(x) \subset \sigma_A(x)$. For the other inclusion, suppose $(x - \lambda 1_A)$ is invertible in B. Then by Theorem 2.8, $(x - \lambda 1_A)^{-1}$ is in the C^*-algebra generated by $x - \lambda 1_A$, i.e. $x - \lambda 1_A$ is invertible in A. Thus $\mathbb{C} \setminus \sigma_B(x) \subset \mathbb{C} \setminus \sigma_A(x)$. So $\sigma_B(x) \supset \sigma_A(x)$. ☐

Corollary 2.10. Let A be a unital C^*-algebra, and $x \in A$ normal . Then $\|\hat{x}\|_\infty = \rho(x)$. [need self-adjoint?]

Proof. Since x is normal, $A(x)$ is commutative (Corollary 2.5). Thus $\mathrm{im}(\hat{x}) = \sigma_{A(x)}(x)$ by Theorem 1.37. By Corollary 2.9, $\sigma_{A(x)}(x) = \sigma_A(x)$. Now by definition,

$$\|\hat{x}\|_\infty = \sup\{|\hat{x}(\ell)| = |\ell(x)| : \ell \in \mathrm{Sp}(A),\ \|\ell\| \leq 1\}.$$

But by Proposition 1.30, it is always true that $\|\ell\| \leq 1$. Hence

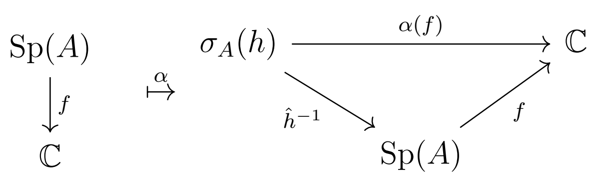
$$\|\hat{x}\|_\infty = \sup\{|\hat{x}(\ell)| = |\ell(x)| : \ell \in \mathrm{Sp}(A)\} = \sup\{|\lambda| : \lambda \in \mathrm{im}(\hat{x}) = \sigma_A(x)\} = \rho(x).$$

☐

Theorem 2.11. Let A be a unital C^*-algebra generated by a single normal element $h \in A$. Then there is an isometric $*$-isomorphism between A and the algebra $C(\sigma_A(h))$, mapping polynomials in h to the same polynomials in $C(\sigma_A(h))$.

Proof. By Corollary 2.5, $A = A(x)$ is commutative. Hence by Theorem 2.7 it is isometrically $*$-isomorphic to the C^*-algebra $C(\mathrm{Sp}(A))$. By Theorem 1.37, $\hat{h} : \mathrm{Sp}(A) \to \sigma_A(h)$ is a homeomorphism. Now define

$$\begin{aligned} \alpha : C(\mathrm{Sp}(A)) &\longrightarrow C(\sigma_A(h)) \\ f &\mapsto f \circ \hat{h}^{-1} \end{aligned}$$



In particular, $\alpha(\hat{h})(\lambda) \coloneqq f \circ \hat{h}^{-1}(\lambda)$, which shows α is an isometric $*$-isomorphism. Hence [work] [ref]

$$\alpha \circ \widehat{(-)} : A \longrightarrow C(\sigma_A(h))$$

is an isometric $*$-isomorphism.

Let's show that this map is how we describe in the statment of the theorem. Let p be a complex polynomial. Then

$$\begin{aligned} (\alpha \circ \widehat{p(h)})(\lambda) =& (\alpha \circ p(\hat{h}))(\lambda) \\ =& (p \circ \alpha(\hat{h}))(\lambda) \\ =& p(\lambda), \end{aligned}$$

where we have used the linearity of α. ☐

Theorem 2.12. Let Ω be a compact Hausdorff space. Then we have the following homeomorphism: $\mathrm{Sp}(C(\Omega)) \cong \Omega$.

Proof. The idea is the following. The map exhibiting the homeomorphism will be the function sending $\omega \in \Omega$ the the evaluation map ϕ_ω which sends $a \in C(\Omega)$ to $a(\omega)$. This map is injective because $C(\Omega)$ separates points of Ω. To show it is surjective, we suppose for the sake of contradiction that for $\ell \in \mathrm{Sp}(A)$ there does not exist $\omega \in \Omega$ such that $\ell = \phi_\omega$. Then for each ω we can construct $b_\omega \in C(\Omega)$ which does not vanish at ω, hence does not vanish in a neighborhood of ω, but $\ell(b_w) = 0$. By compactness we can cover Ω with finitely many such neighborhoods, and define a continuous function x to be the sum of the finitely many corresponding b_ω. We can make this function positive everywhere on Ω. We then derive a $0 = 1$ contradiction using the fact that $\ell(x) = 0$ but $\ell(xx^{-1}) = \ell(1_A) = 1$.

For each $\omega \in \Omega$, define the evaluation maps

$$\begin{aligned}\phi_\omega : C(\Omega) &\longrightarrow \mathbb{C}\\ a &\mapsto a(\omega)\end{aligned}$$

One checks ϕ_ω is a character, so $\phi_\omega \in \mathrm{Sp}(C(\Omega))$. Note if $\phi_{\omega_1} = \phi_{\omega_2}$ then $a(\omega_1) = a(\omega_2)$ for all $a \in C(\Omega)$. By , $C(\Omega)$ separates points of Ω, so $a(\omega_1) = a(\omega_2)$ for all $a \in C(\Omega)$ implies $\omega_1 = \omega_2$. Thus $\omega \mapsto \phi_\omega$ is injective. work ref

To show it is surjective, let $\ell \in \mathrm{Sp}(C(\mathrm{sp}(A)))$ and suppose for the sake of contradiction that does not exist $\omega \in \Omega$ such that $\ell = \phi_\omega$. Then $\ell - \phi_\omega \neq 0$ for all $\omega \in \Omega$. So far each $\omega \in \Omega$, there exists $a_w \in C(\Omega)$ such that $\ell(a_\omega) - \phi(a_\omega) \neq 0$, i.e. $\ell(a_\omega) \neq a_\omega(\omega)$. Write $b_\omega :- a_\omega - \ell(a_\omega)1_A$. Then $b_\omega \neq 0$ (since $b_\omega(\omega) \neq 0$) but $\ell(b_\omega) = 0$.

Since $b_\omega \in C(\Omega)$, there exists a neighborhood N_ω of ω such that b_ω does not vanish on N_ω. Varying ω over Ω we get an open cover $\{N_\omega : \omega \in \Omega\}$ of Ω. Since Ω is compact, we get a finite subcover $\{N_{\omega_1}\}$. Write

$$x = |b_{\omega_1}|^2 + \cdots + |b_{\omega_k}|^2.$$

Then $x \in C(\Omega)$ and $x(\omega) > 0$ for all $\omega \in \Omega$. Also

$$\begin{aligned}\ell(x) =&\ell(b^*_{\omega_1}) + \cdots + \ell(b^*_{\omega_k} b_{\omega_k})\\ =&\ell(b^*_{\omega_1})\ell(b_{\omega_1}) + \cdots + \ell(b^*_{\omega_k})\ell(b_{\omega_k})\\ =&0,\end{aligned}$$

since $\ell(b_{\omega_j}) = 0$ for all $j = 1, \ldots, k$. So $x \in \ker(\ell)$. But $x > 0$ implies x^{-1} exists in $C(\Omega)$, and we reach a contradiction: ref

$$0 = \ell(x)\ell(x^{-1}) = \ell(xx^{-1}) = \ell(1_A) = 1.$$

This shows surjectivity.

To show this is a homeomorphism, since Ω is compact by assumption and $\mathrm{Sp}(A)$ is compact by Proposition 1.35, it sufficies to show $\phi_{(-)}$ is continuous . Suppose a net $(w_\alpha) \subset \Omega$ converges ref to ω. Then for each $x \in C(\Omega)$,

$$\phi_{\omega_\alpha}(x) = x(\omega_\alpha) \to x(\omega) = \phi_\omega(x)$$

by the continuity of x. This $\phi_{\omega_\alpha} \to \phi_\omega$ and we're done. ☐

Theorem 2.13 (unitization)**.** Let A be a C^*-algebra without unit. Then A is isometrically $*$-isomorphic to a C^*-algebra of codimension 1 in a unital C^*-algebra.

Proof. We define the unital C^*-algebra as follows:

as a set	$\tilde{A} = A \oplus \mathbb{C}$
involution	$a \oplus \lambda \mapsto a^* \oplus \overline{\lambda}$
multiplication	$(a \oplus \lambda)(b \oplus \mu) = (ab + \lambda b + \mu a) \oplus \lambda\mu$
norm	$\lVert a \oplus \lambda \rVert = \sup\{\lVert ax + \lambda x \rVert : x \in A,\ \lVert x \rVert \leq 1\}$
unit	$0 \oplus 1$

We check this works. ☐ check

Theorem 2.14. The norm on a C^*-algebra is unique.

Proof. Suppose n_1 and n_2 are norms on A making A a C^*-algebra. Without loss of generality we may assume A is unital, by Theorem 2.13. Let $h \in A$ be self-adjoint. Then ref

$$n_1(h) = \lVert \hat{h} \rVert_\infty = \sup\{|\lambda| : \lambda \in \sigma(h)\} = n_2(h),$$

since inverses are defined algebraically (i.e. independent of norm).

In general, for any $x \in A$,

$$n_1(x)^2 = n_1(x^*x) = n_2(x^*x) = n_2(x)^2$$

since x^*x is self-adjoint. ☐

Corollary 2.15. Let A be a non-unital C^*-algebra. Then unitization commutes with taking the subalgebra generated by a:

$$\begin{array}{ccc} A & \longrightarrow & A(x) \\ \downarrow & & \downarrow \\ \tilde{A} & \longrightarrow & \tilde{A}(x) = \widetilde{A(x)} \end{array}$$

Theorem 2.16. Let A be a commutative C^*-algebra without unit. Then there exists a locally compact, non-compact, Hausdorff space X such that A is isometrically $*$-isomorphic to $C_0(X)$, the C^*-algebra of continuous $\mathbb{C}$-valued functions on X vanishing at infinity.

Proof. Let $K = \mathrm{Sp}(\tilde{A})$. By Proposition 1.35, K is a compact Hausdorff space. By Theorem 2.7, $\tilde{A} \simeq C(K)$, and so A is isometrically $*$-isomorphic to a C^*-subalgebra of $C(K)$ via the Gelfand transform on $\tilde{A}$.

Let $\kappa_0 \in K$ be defined as follows:

$$\kappa_0(a) = \begin{cases} 0 & a \in A \subset \tilde{A} \\ 1 & a = 1_A \end{cases}.$$

So for any $a \in A$, we have $\hat{a}(\kappa_0) = 0$, so the image of A under $\widehat{(-)}$ consists of functions in $C(K)$ which vanish at $\kappa_0 \in K$. check is char

Conversely, suppose $f \in C(K)$ is such that $f(\kappa_0) = 0$. Let $x \in \tilde{A}$ be such that $\hat{x} = f$. By the construction of unitization, we can write $x = a + \mu 1_A$ for some $a \in A$ and $\mu \in \mathbb{C}$. Then

$$\begin{aligned} f(\kappa_0) = 0 &\Rightarrow \hat{x}(\kappa_0) = 0 \\ &\Rightarrow \kappa_0(x) = 0 \\ &\Rightarrow \kappa_0(a) + \mu\kappa_0(1_A) = 0 \\ &\Rightarrow \mu = 0, \end{aligned}$$

since $\kappa_0(a) = 0$ for all $a \in A$. Thus $x \in A$, so the Gelfand transform maps A onto the subalgebra of $C(K)$ consisting of those functions which vanish at κ_0.

Let $X = K \setminus \{k\kappa_0\}$. Then X is locally compact and the map $g \mapsto g|_X$ is an isometric $*$-isomorphism ref

$$A = \{g \in C(K) : g(\kappa_0) = 0\} \longrightarrow C_0(X)$$

It remains to show X is not compact. If it were, κ_0 would be an isolated point of K and the element $e \in A \subset \tilde{A}$ corresponding to the continuous function ref

$$\hat{e}(\kappa) = \begin{cases} 0 & \kappa = \kappa_0 \\ 1 & \text{otherwise} \end{cases}$$

would be a unit for A, a contradiction. □

3 Gelfand-Naimark theorems

The following is stated as Theorem A.2. in [1]:

Theorem 3.1.

1. Let A be a unital Banach algebra over $\mathbb{C}$. If $a \in A$, then $\sigma(a)$ is nonempty.
2. Let A be a unital algebra over $\mathbb{C}$ of countable dimension. If $a \in A$, then $\sigma(a)$ is nonempty. Furthermore, a is nilpotent if and only if $\sigma(a) = \{0\}$.

Proof of 1. Suppose $\sigma(a) = \emptyset$. Then the function

$$\begin{aligned} R : \mathbb{C} &\to A \\ \lambda &\mapsto (a - \lambda 1)^{-1} \end{aligned}$$

is holomorphic, non-constant, and bounded. But this contradicts Liouville's theorem for Banach space valued functions. □

Proof of 2. Suppose $\sigma(a) = \emptyset$. Then $(T - \lambda 1)^{-1}$ exists for all $\lambda \in \mathbb{C}$.

Claim 3.2. There is an injective, linear homomorphism $\phi : \mathbb{C}(X) \to A$ sending $X \mapsto T$.

Proof. Any element in $\mathbb{C}(X)$ may be expressed as $\frac{p(X)}{q(X)}$, where $p(X), q(X) \in \mathbb{C}[X]$. It is clear that $p(X) \mapsto p(T)$ is injective and linear, and it remains to show we can compatibly map $\frac{1}{q(X)}$. By the fundamental theorem of algebra we can write $q(X) = (X - \lambda_1) \cdots (X - \lambda_n)$. By assumption, $(T - \lambda 1)^{-1}$ exists, so map $\frac{1}{q(X)}$ to $(T - \lambda_1 1)^{-1} \cdots (T - \lambda_n 1)^{-1}$. □

By the uniqueness of partial fraction decomposition, the set

$$\left\{ \frac{1}{X - \lambda} \right\}_{\lambda \in \mathbb{C}}$$

are linearly independent. Since ϕ is injective and linear, so are their images under ϕ, i.e. the $\{(T - \lambda 1)^{-1}\}$ are linearly independent. But then this would provide an uncountable basis for A, contradicting our assumption. □

References

[1] Masoud Khalkhali. *Basic noncommutative geometry*. European Mathematical Society, 2013.